Exercise 1. Only the integrability when $|x| \to \infty$ matters, and when $|x| \to \infty$, the function u is equivalent to

$$v(x) = \frac{1}{2}\frac{1}{|x|^\alpha \log|x|}.$$

Therefore, if $p > \dfrac{1}{\alpha}$, we have

$$\int_e^\infty |v(x)|^p dx \leq \frac{1}{2^p}\int_e^\infty \frac{dx}{x^{p\alpha}} = \frac{1}{2^p}\frac{1}{p\alpha - 1}\frac{1}{e^{p\alpha-1}} < \infty.$$

And since u and v are even, we deduce that

$$\int_{\mathbb{R}} |u(x)|^p dx < \infty.$$

For $p = \dfrac{1}{\alpha}$, we have

$$\int_e^\infty |v(x)|^p = \frac{1}{2^p}\int_e^\infty \frac{dx}{x\log^p(x)} = \frac{1}{2^p}\left[-\frac{1}{p-1}\frac{1}{\log^{p-1}(x)}\right]_e^\infty = \frac{1}{2^p(p-1)} < \infty$$

since $0 < \alpha < 1$. Now, we have

$$u'(x) = -\frac{\alpha}{2}\frac{1}{(1+x^2)^{\frac{\alpha}{2}+1}\log(2+x^2)} - \frac{2}{(1+x^2)^{\frac{\alpha}{2}}(2+x^2)\log^2(2+x^2)}.$$

Therefore, as $|x| \to \infty$, u is bounded (up to a constant) by

$$\frac{1}{|x|^{\alpha+1}\log|x|}$$

and the previous computation apply *a fortiori* to show that $u' \in L^p(\mathbb{R})$ for all $\dfrac{1}{\alpha} \leq p < \infty$. On the other hand, since for all $0 < \beta < 1$ and $\gamma > 0$, we have

$$\frac{1}{|x|^\beta \log^\gamma(|x|)} \notin L^1(\mathbb{R}),$$

we deduce that $u \notin L^q(\mathbb{R})$ for all $0 < q < \dfrac{1}{\alpha}$.

Exercise 2. This is easy to see since for all $x, y \in [a,b]$, we have

$$u(x) - u(y) = \int_y^x u'(t)dt.$$

Therefore, by additivity of the integral, we have by the triangle inequality

$$\sum_{i=1}^m |u(b_i) - u(a_i)| = \sum_{i=1}^m \left|\int_{a_i}^{b_i} u'(t)dt\right| \leq \sum_{i=1}^m \int_{a_i}^{b_i} |u'(t)|dt.$$

Since $\displaystyle\int_a^b |u'(t)|dt < \infty$, a classical property of the Lebesgue integral shows that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all disjoint intervals $I_1, \cdots, I_m \subset [a,b]$, we have

$$\sum_{i=1}^m \mathscr{L}^1(I_i) < \delta \implies \sum_{i=1}^m \int_{I_i} |u'(t)|dt < \varepsilon.$$

and the propriety follows.

Exercise 3. 1. First, assume that $u \in C^1([0,1])$. Then the boundedness holds since $x \mapsto \frac{u(x)}{x}$ is bounded. Since $u(0) = 0$, we write

$$u(x) = \int_0^x u'(t)dt.$$

Therefore, we have

$$\int_0^1 \frac{|u(x)|^p}{|x|^p}dx = \int_0^1 \left|\int_0^x u'(t)dt\right|^p \frac{dx}{x^p}.$$

Since

$$\frac{d}{dx}\left(\int_0^x u'(t)dt\right) = u'(x),$$

it is easy to show that $x \mapsto \left|\int_0^x u'(t)dt\right|^p$ is a C^1 function and that

$$\frac{d}{dx}\left|\int_0^x u'(t)dt\right|^p = p\,u'(x)\left|\int_0^x u'(t)dt\right|^{p-1}.$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^1 \left|\int_0^x u'(t)dt\right|^p \frac{dx}{x^p} &= \left[-\frac{1}{p-1}\frac{1}{x^{p-1}}\left|\int_0^x u'(t)dt\right|^p\right]_0^1 + \frac{p}{p-1}\int_0^1 u'(x)\left|\int_0^x u'(t)dt\right|^{p-1}\frac{dx}{x^{p-1}} \\ &= -\frac{1}{p-1}|u(1)| + \frac{p}{p-1}\int_0^1 u'(x)\left|\int_0^x u'(t)dt\right|^{p-1}\frac{dx}{x^{p-1}} \\ &\leq \frac{p}{p-1}\int_0^1 u'(x)\left|\int_0^x u'(t)dt\right|^{p-1}\frac{dx}{x^{p-1}}.\end{aligned}$$

Since $u' \in C^0([0,1])$, notice that as $x \to 0$, we have

$$\int_0^x u'(t)dt = O(x),$$

and as a consequence,

$$\frac{1}{x^{p-1}}\left|\int_0^x u'(t)dt\right|^p = O(x)$$

and the integration by parts formula holds. Now, by Hölder's inequality, we have

$$\int_0^1 u'(x)\left|\int_0^x u'(t)dt\right|^{p-1}\frac{dx}{x^{p-1}} \leq \|u'\|_{\mathrm{L}^p([0,1])}\left(\int_0^1 \frac{|u(x)|^p}{|x|^p}dx\right)^{\frac{p-1}{p}},$$

which yields

$$\int_0^1 \frac{|u(x)|^p}{|x|^p}dx \leq \frac{p}{p-1}\|u'\|_{\mathrm{L}^p([0,1])}\left(\int_0^1 \frac{|u(x)|^p}{|x|^p}dx\right)^{\frac{p-1}{p}}$$

and

$$\left\|\frac{u(x)}{x}\right\|_{\mathrm{L}^p([0,1])} \leq \frac{p}{p-1}\|u'\|_{\mathrm{L}^p([0,1])}$$

and the expected inequality follows (assuming that $u \neq 0$; otherwise, the inequality holds trivially).

2. Since $u \in C^0([0,1])$ thanks to the Sobolev embedding, if $u(0) \neq 0$, then there exists $\delta > 0$ and $\varepsilon > 0$ such that $|u(x)| \geq \varepsilon$ for all $0 \leq x < \delta$, which implies that

$$\int_0^\delta \frac{|u(x)|^p}{|x|^p}dx \geq \int_0^\delta \frac{\varepsilon^p}{x^p}dx = \infty.$$

3. u is a bounded function, which implies that $u \in L^1([0,1])$. Furthermore, we have

$$u'(x) = -\frac{1}{x\left(1+\log\left(\frac{1}{x}\right)\right)^2}.$$

Therefore, u' is integrable locally on $]0,1]$, and since

$$\int_0^{\frac{1}{e}} \frac{dx}{x\log^2\left(\frac{1}{x}\right)} = \left[\frac{1}{\log\left(\frac{1}{x}\right)}\right]_0^{\frac{1}{e}} = 1 < \infty,$$

we deduce that $u \in W^{1,1}(]0,1[)$. On the other hand, we have

$$\int_0^{\frac{1}{e}} \frac{dx}{x\log\left(\frac{1}{x}\right)} = \left[-\log\log\left(\frac{1}{x}\right)\right]_0^{\frac{1}{e}} = \infty,$$

and since $x \mapsto \dfrac{u(x)}{x} = \dfrac{1}{x\left(1+\log\left(\frac{1}{x}\right)\right)}$ is equivalent to $x \mapsto \dfrac{1}{x\log\left(\frac{1}{x}\right)}$ as $x \mapsto 0$, we deduce that $x \mapsto \dfrac{u(x)}{x} \notin L^1([0,1])$.

Exercise 4. 1. Integrating by parts, if $u \in W^{1,1}(\Omega)$, we have for all $\varphi \in C_c^1(\Omega, \mathbb{R}^d)$

$$\int_\Omega u \operatorname{div}\varphi\, dx = -\int_\Omega \nabla u \cdot \varphi\, dx,$$

which implies that

$$\int_\Omega u \operatorname{div}\varphi\, dx \leq \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \int_\Omega |\nabla u| dx,$$

and

$$\sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx : \varphi \in C_c^1(\Omega,\mathbb{R}^d), \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \leq 1\right\} \leq \|\nabla u\|_{\mathrm{L}^1(\Omega)} < \infty$$

by hypothesis.

2. Integrating by parts, we have for all $\varphi \in C_c^1(]-1,1[,\mathbb{R})$

$$\int_{-1}^1 H(x)\varphi'(x)dx = \int_0^1 \varphi'(x)dx = -\varphi(0)$$

which shows that

$$\sup\left\{\int_{-1}^1 H(x)\varphi'(x) : \varphi \in C_c^1(]-1,1[,\mathbb{R}), \|\varphi\|_{\mathrm{L}^\infty([-1,1])} \leq 1\right\}$$
$$= \sup\left\{-\varphi(0) : \varphi \in C_c^1(]-1,1[,\mathbb{R}), \|\varphi\|_{\mathrm{L}^\infty([-1,1])} \leq 1\right\} = 1 < \infty.$$

Notice that we need only prove the inequality, but it is clear that equality holds in this identity (by choosing the appropriate bump function as constructed in the lecture notes). However, as $H \notin C^0([-1,1])$, the Sobolev embedding $W^{1,1}(]-1,1[) \hookrightarrow C^0([-1,1])$ shows that $H \notin W^{1,1}(]-1,1[)$.

3. It is easy to see that u is continuous outside 0, and at 0, we simply use the fact that sin is bounded to see that it is also continuous at $x = 0$, and therefore continuous everywhere. The issue is that the derivative blows up at 0. The idea is to use the rapidly oscillatory behaviour of $\sin(1/x)$ near 0 to get arbitrarily large contributions in the integral

$$\int_{-1}^{1} u(x)\varphi'(x)dx.$$

For all $n \geq 1$, define the piecewise linear function such that for all $0 \leq k \leq n$

$$\varphi_n(x) = \begin{cases} 0 & \text{for all } x \in \left[\frac{1}{8k+9}, \frac{1}{8k+7}\right] \\ \frac{(8k+7)(8k+5)}{2}\left(\frac{1}{8k+7} - x\right) & \text{for all } x \in \left[\frac{1}{8k+7}, \frac{1}{8k+5}\right] \\ -1 & \text{for all } x \in \left[\frac{1}{8k+5}, \frac{1}{8k+3}\right] \\ \frac{(8k+3)(8k+1)}{2}\left(x - \frac{1}{8k+1}\right) & \text{for all } x \in \left[\frac{1}{8k+3}, \frac{1}{8k+1}\right] \end{cases}$$

and φ_n vanishes otherwise. Notice that $-1 \leq \varphi_n \leq 0$ and that $\varphi_n \in W_0^{1,\infty}(]-1,1[)$. Furthermore, for all $\dfrac{1}{8k+7} \leq x \leq \dfrac{1}{8k+5}$, we have

$$\frac{\pi}{4x} \in \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] + 2\pi k,$$

which implies that

$$-1 \leq \sin\left(\frac{\pi}{4x}\right) \leq -\frac{\sqrt{2}}{2},$$

which shows by an integration by parts that

$$\int_{\frac{1}{8k+7}}^{\frac{1}{8k+5}} x\sin\left(\frac{\pi}{4x}\right)\varphi_n'(x)dx \geq \frac{\sqrt{2}}{2}\frac{1}{8k+7},$$

and likewise

$$\int_{\frac{1}{8k+3}}^{\frac{1}{8k+1}} x\sin\left(\frac{\pi}{4x}\right)\varphi_n'(x)dx \geq \frac{\sqrt{2}}{2}\frac{1}{8k+3}.$$

Therefore, we deduce that for all $n \in \mathbb{N}$, we have

$$\int_{-1}^{1} u(x)\varphi_n'(x) \geq \frac{\sqrt{2}}{2}\sum_{k=0}^{n}\left(\frac{1}{8k+3} + \frac{1}{8k+7}\right) \underset{n\to\infty}{\longrightarrow} \infty.$$

Now, we need only approximate each φ_n by a function $\psi_n \in C_c^1(]-1,1[)$ such that

$$\|\psi_n - \varphi_n\|_{\mathrm{W}^{1,1}(]-1,1[)} \leq 1 \qquad \text{and} \quad \|\psi_n\|_{\mathrm{L}^\infty(]-1,1[)} \leq 1$$

which yields

$$\begin{aligned}V(u,]-1,1[) &\geq \int_{-1}^{1} u(x)\psi_n'(x)dx = \int_{-1}^{1} u(x)\varphi_n'(x)dx + \int_{-1}^{1} u(x)\left(\psi_n'(x) - \varphi_n'(x)\right)dx \\ &\geq \int_{-1}^{1} u(x)\varphi_n'(x)dx - \|u\|_{\mathrm{L}^\infty([-1,1])}\|\psi_n - \varphi_n\|_{\mathrm{W}^{1,1}(]-1,1[)} \\ &\geq \int_{-1}^{1} u(x)\varphi_n'(x)dx - \|u\|_{\mathrm{L}^\infty([-1,1])} \underset{n\to\infty}{\longrightarrow} \infty.\end{aligned}$$

Since $-1 \leq \varphi_n \leq 0$, up to replacing φ_n by $\dfrac{1}{2}\varphi_n$, a standard convolution will yield the expected bound $\|\psi_n\|_{\mathrm{L}^\infty([-1,1])} \leq 1$ on the approximation and the proof is complete.